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Klein, A ; Jetzer, P

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# Spin effects in the phasing of gravitational waves from binaries on eccentric orbits

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We compute here the spin-orbit and spin-spin couplings needed for an accurate computation of the phasing of gravitational waves emitted by comparable-mass binaries on eccentric orbits at the second post-Newtonian (PN) order. We use a quasi-Keplerian parametrization of the orbit free of divergencies in the zero eccentricity limit. We find that spin-spin couplings induce a residual eccentricity for coalescing binaries at 2PN, of the order of  $10^{-4}$ - $10^{-3}$  for supermassive black hole binaries in the LISA band. Spin-orbit precession also induces a non-trivial pattern in the evolution of the eccentricity, which could help to reduce the errors on the determination of the eccentricity and spins in a gravitational wave measurement.

## I. INTRODUCTION

Up to now most studies concerning gravitational wave emission from binaries have been done assuming circular orbits (see e.g. [1, 2] and references therein). Numerical studies of the formation of binaries and of their subsequent development suggest instead that the orbits could be eccentric [3, 4]. It is thus of relevance to take the eccentricity into account when one computes the waveform emitted by such a system. This problem has not yet been fully explored, mainly due to its complexity, and here as a further step we derive the equations which govern the time evolution of the orbital parameters, and in particular the eccentricity, including the spin-orbit and the spin-spin couplings needed for an accurate computation in the post-Newtonian approximation up to 2PN of the phasing of the gravitational waves emitted by binary systems, with components of comparable mass.

The PN approximation is valid when the two object forming the binary are sufficiently separated. The issue of estimating the limit of its validity has been tackled with different methods: comparisons between post-Newtonian templates and results from numerical relativity, for non-spinning binaries on quasi-circular orbits [5]; and comparisons between different post-Newtonian waveforms for spinning and non-spinning binaries on quasi-circular orbits [6, 7]. All of these studies found a remarkable reliability of the PN approximation up to separations as small as the innermost stable circular orbit,  $r = 6GM/c^2$ . It is not clear if that still holds for eccentric binaries, and this must be answered by extending these comparisons to such systems, but one can trust that for low enough eccentricities, the post-Newtonian approximation is reliable up to the end of the inspiral.

We also derive a quasi-Keplerian parametrization of the orbit free of divergencies in the zero eccentricity limit, and find that spin-spin couplings induce a residual eccentricity of 2PN order after the orbit has been circularized by gravitational wave emission.

We then solve the equations which govern the evolu-

tion of the eccentricity and the mean motion for different values of the masses and spins as a function of the initial eccentricity.

## II. KEPLER EQUATIONS AND EVOLUTION OF THE MEAN MOTION AND OF THE ECCENTRICITY

As spin-orbit couplings appear at 1.5PN order and spin-spin couplings at 2PN order, it is sufficient to consider only the Newtonian and spin-coupling term in the equations of motion. For simplicity, we will use a system of units where  $G = c = M = 1$ , where  $M$  is the total mass of the system. We start from the generalized Lagrangian in the center of mass frame used in [8, 9]:

$$\mathcal{L} = \frac{\nu}{2} \mathbf{v}^2 + \frac{\nu}{r} + \frac{\nu}{2} (\mathbf{v} \times \mathbf{a}) \cdot \boldsymbol{\xi} - \frac{2\nu}{r^3} (\mathbf{x} \times \mathbf{v}) \cdot (\boldsymbol{\zeta} + \boldsymbol{\xi}) + \frac{1}{r^3} \mathbf{S}_1 \cdot \mathbf{S}_2 - \frac{3}{r^5} (\mathbf{x} \cdot \mathbf{S}_1) (\mathbf{x} \cdot \mathbf{S}_2), \quad (1)$$

where

$$\nu = m_1 m_2, \quad (2)$$

$$r = |\mathbf{x}|, \quad (3)$$

$$\boldsymbol{\zeta} = \mathbf{S}_1 + \mathbf{S}_2, \quad (4)$$

$$\boldsymbol{\xi} = \frac{m_2}{m_1} \mathbf{S}_1 + \frac{m_1}{m_2} \mathbf{S}_2. \quad (5)$$

The equations of motion are

$$\mathbf{p}^i = \frac{\partial \mathcal{L}}{\partial \mathbf{v}^i} - \frac{d}{dt} \mathbf{s}^i, \quad (6)$$

$$\frac{d\mathbf{p}^i}{dt} = \frac{\partial \mathcal{L}}{\partial \mathbf{x}^i}, \quad (7)$$

where  $\mathbf{s}^i = \partial \mathcal{L} / \partial \mathbf{a}^i$ .

We can solve them order by order, which gives at 2PN order

$$\mathbf{p} = \nu \mathbf{v} + \frac{\nu}{r^3} \mathbf{x} \times (2\boldsymbol{\zeta} + \boldsymbol{\xi}), \quad (8)$$

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$$\begin{aligned}
\mathbf{a} = & -\frac{\mathbf{x}}{r^3} + \frac{\mathbf{x} \cdot \mathbf{v}}{r^5} \mathbf{x} \times (6\boldsymbol{\zeta} + 3\boldsymbol{\xi}) \\
& - \frac{1}{r^3} \mathbf{v} \times (4\boldsymbol{\zeta} + 3\boldsymbol{\xi}) + \frac{\mathbf{x}}{r^5} (\mathbf{x} \times \mathbf{v}) \cdot (6\boldsymbol{\zeta} + 6\boldsymbol{\xi}) \\
& - \frac{3\mathbf{x}}{\nu r^5} \mathbf{S}_1 \cdot \mathbf{S}_2 - \frac{3}{\nu r^5} [(\mathbf{x} \cdot \mathbf{S}_2) \mathbf{S}_1 + (\mathbf{x} \cdot \mathbf{S}_1) \mathbf{S}_2] \\
& + \frac{15\mathbf{x}}{\nu r^7} (\mathbf{x} \cdot \mathbf{S}_1) (\mathbf{x} \cdot \mathbf{S}_2). \tag{9}
\end{aligned}$$

The reduced energy and reduced orbital angular momentum are given by

$$\begin{aligned}
\mathbf{J} = & \frac{1}{\nu} (\mathbf{x} \times \mathbf{p} + \mathbf{v} \times \mathbf{s}) \\
= & \mathbf{x} \times \mathbf{v} + \frac{1}{r^3} \mathbf{x} \times [\mathbf{x} \times (2\boldsymbol{\zeta} + \boldsymbol{\xi})] - \frac{1}{2} \mathbf{v} \times (\mathbf{v} \times \boldsymbol{\xi}), \tag{10}
\end{aligned}$$

$$\begin{aligned}
E = & \frac{1}{\nu} (\mathbf{p} \cdot \mathbf{v} + \mathbf{s} \cdot \mathbf{a} - \mathcal{L}) \\
= & \frac{1}{2} \mathbf{v}^2 - \frac{1}{r} + \frac{1}{r^3} (\mathbf{x} \times \mathbf{v}) \cdot \boldsymbol{\xi} \\
& - \frac{1}{\nu r^3} \mathbf{S}_1 \cdot \mathbf{S}_2 + \frac{3}{\nu r^5} (\mathbf{x} \cdot \mathbf{S}_1) (\mathbf{x} \cdot \mathbf{S}_2). \tag{11}
\end{aligned}$$

The magnitude of  $\mathbf{J}$  is not constant along an orbit [10]. Indeed, due to spin-spin interactions, both spin vectors undergo a precessional motion and thus, from the conservation of the total angular momentum, it follows that  $\mathbf{J}$  changes at the 2PN order. If we denote its angular average (with respect to the true anomaly  $v$ , defined later) by  $L$ , and define  $A = \sqrt{1 + 2EL^2}$ , we get

$$\begin{aligned}
J = & L - \frac{1}{2\nu L^3} \left| \hat{\mathbf{J}} \times \mathbf{S}_1 \right| \left| \hat{\mathbf{J}} \times \mathbf{S}_2 \right| \{2A \cos(v - 2\psi) \\
& + (3 + 2A \cos v) \cos[2(v - \psi)]\} = \\
= & L - \frac{\gamma_2}{2L^3} \{3A \cos(v - 2\psi) \\
& + 3 \cos[2(v - \psi)] + A \cos(3v - 2\psi)\}, \tag{12}
\end{aligned}$$

$$\gamma_2 = \frac{1}{\nu} \left| \hat{\mathbf{J}} \times \mathbf{S}_1 \right| \left| \hat{\mathbf{J}} \times \mathbf{S}_2 \right|, \tag{13}$$

where we defined  $\psi$  to be the angle subtended by the bisector of the projections of  $\mathbf{S}_i$  in the plane of motion and the periastron line.

We can find a quasi-Keplerian solution to these equations, as (see the appendix)

$$r = a(1 - e_r \cos u) + f_r \cos[2(v - \psi)], \tag{14}$$

$$\phi = (1 + k)v + f_{\phi,1} \sin(v - 2\psi) + f_{\phi,2} \sin[2(v - \psi)], \tag{15}$$

$$v = 2 \arctan \left( \sqrt{\frac{1 + e_\phi}{1 - e_\phi}} \tan \frac{u}{2} \right), \tag{16}$$

$$l = n(t - t_0) = u - e_t \sin u, \tag{17}$$

where  $(r, \phi)$  is a polar coordinate system in the plane of motion,  $n$  is the mean motion,  $u$ ,  $v$ , and  $l$  are the eccentric, true, and mean anomalies,  $a$  is the semi-major axis,

$e_t$ ,  $e_r$ , and  $e_\phi$  are eccentricities,  $k$  accounts for perihelion precession, and the  $f_i$  are constants.

This parametrization is different from the one found in [11], which suffered from an apparent singularity in the limit  $e \rightarrow 0$  (all three eccentricities tend together towards zero). This singularity was due to the fact that the authors used as a definition of the eccentric anomaly, denoted in their paper by  $\xi$ ,

$$r(\xi) = \frac{1}{2} [r_{\max} + r_{\min} - (r_{\max} - r_{\min}) \cos \xi], \tag{18}$$

which leads to Eq. (14) with  $f_r = 0$ . The zero eccentricity limit of the equations of motion  $r(\phi)$  and  $\dot{\phi}$  leads to  $r = \bar{r} + \delta r \cos[2(\phi - \psi)]$  (see the appendix). If  $f_r = 0$  in Eq. (14), this angular dependence must come from the change of variables  $u(\phi)$ . To cancel the  $e_r = O(e)$  factor in front of  $\cos(u)$  so that the angular dependence does not vanish in the zero eccentricity limit, the function  $u(\phi)$  must be of order  $O(e^{-1})$ . This is the origin of the apparent singularity in the quasi-Keplerian parametrization found in [11].

Our parametrization has the advantage of being free from singularities in the zero eccentricity limit, so that the latter can be more transparently studied. Note, however, that the periastron line (defined by the equation  $u = v = 2p\pi$ ,  $p \in \mathbb{Z}$ ) does no longer correspond to  $r = r_{\min}$ .

The mean motion and time eccentricity are given, in terms of  $E$  and  $L$ , as

$$n = (-2E)^{3/2}, \tag{19}$$

$$e_t^2 = A^2 + \frac{E}{L} \beta (8, 6 - 2A^2) + 2 \frac{E}{L^2} \gamma_1, \tag{20}$$

where

$$\beta(a, b) = \hat{\mathbf{J}} \cdot (a\boldsymbol{\zeta} + b\boldsymbol{\xi}), \tag{21}$$

$$\gamma_1 = \frac{1}{\nu} \left[ \mathbf{S}_1 \cdot \mathbf{S}_2 - 3 \left( \hat{\mathbf{J}} \cdot \mathbf{S}_1 \right) \left( \hat{\mathbf{J}} \cdot \mathbf{S}_2 \right) \right]. \tag{22}$$

We can invert these relations and find  $E$  and  $L$  as functions of the post-Newtonian parameter  $x = n^{2/3}$  and the eccentricity  $e = e_t$ . These are

$$E = -\frac{x}{2}, \tag{23}$$

$$L = \frac{\sqrt{1 - e^2}}{x^{1/2}} \left[ 1 - \frac{x^{3/2} \beta(4, 3 - e^2)}{2(1 - e^2)^{3/2}} - \frac{x^2 \gamma_1}{2(1 - e^2)^2} \right]. \tag{24}$$

These allow us to express the constant parameters of the quasi-Keplerian motion as

$$a = x^{-1} \left[ 1 + \frac{x^{3/2} \beta(2, 1)}{\sqrt{1 - e^2}} + \frac{x^2 \gamma_1}{2(1 - e^2)} \right], \tag{25}$$

We can now use the results from [10, 12], where the orbit averages of  $dE/dt$  and  $dL/dt$  due to the emission of gravitational waves were computed:

$$k = -\frac{x^{3/2}\beta(4, 3)}{(1-e^2)^{3/2}} - \frac{3x^2\gamma_1}{2(1-e^2)^2}, \quad (26)$$

$$e_r = e \left[ 1 - \frac{x^{3/2}\beta(2, 1)}{\sqrt{1-e^2}} - \frac{x^2\gamma_1}{2(1-e^2)} \right], \quad (27)$$

$$e_\phi = e \left[ 1 - \frac{x^{3/2}\beta(2, 2)}{\sqrt{1-e^2}} - \frac{x^2\gamma_1}{(1-e^2)} \right], \quad (28)$$

$$f_r = -\frac{x}{2(1-e^2)}\gamma_2, \quad (29)$$

$$f_{\phi,1} = -\frac{ex^2}{(1-e^2)^2}\gamma_2, \quad (30)$$

$$f_{\phi,2} = -\frac{x^2}{4(1-e^2)^2}\gamma_2. \quad (31)$$

where

$$\dot{E}_N = -\frac{(-2E)^{3/2}}{15L^7} (96 + 292A^2 + 37A^4), \quad (34)$$

$$\dot{E}_{SO} = \frac{(-2E)^{3/2}}{10L^{10}} \beta (2704 + 7320A^2 + 2490A^4 + 65A^6, 1976 + 5096A^2 + 1569A^4 + 32A^6), \quad (35)$$

$$\begin{aligned} \dot{E}_{SS} = \frac{(-2E)^{3/2}}{960L^{11}} [ & 2\sigma(42\,048 + 154\,272A^2 + 75\,528A^4 + 3084A^6, 124\,864 + 450\,656A^2 + 215\,544A^4 + 8532A^6, \\ & 131\,344A^2 + 127\,888A^4 + 7593A^6) - \tau(448 + 4256A^2 + 3864A^4 + 252A^6, 64 + 608A^2 + 552A^4 + 36A^6, \\ & 16A^2 + 80A^4 + 9A^6) ], \end{aligned} \quad (36)$$

$$\dot{L}_N = -\frac{4(-2E)^{3/2}}{5L^4} (8 + 7A^2), \quad (37)$$

$$\dot{L}_{SO} = \frac{(-2E)^{3/2}}{15L^7} \beta (2264 + 2784A^2 + 297A^4, 1620 + 1852A^2 + 193A^4), \quad (38)$$

$$\begin{aligned} \dot{L}_{SS} = \frac{(-2E)^{3/2}}{20L^8} [ & 2\sigma(552 + 996A^2 + 132A^4, 1616 + 2868A^2 + 381A^4, 894A^2 + 186A^4) \\ & - (8 + 24A^2 + 3A^4) \tau(2, 1, 0) ], \end{aligned} \quad (39)$$

$$\sigma(a, b, c) = \frac{1}{\nu} \left[ a\mathbf{S}_1 \cdot \mathbf{S}_2 - b(\hat{\mathbf{J}} \cdot \mathbf{S}_1)(\hat{\mathbf{J}} \cdot \mathbf{S}_2) + c|\hat{\mathbf{J}} \times \mathbf{S}_1||\hat{\mathbf{J}} \times \mathbf{S}_2| \cos 2\psi \right], \quad (40)$$

$$\tau(a, b, c) = \sum_{i=1}^2 \frac{1}{m_i^2} \left[ a\mathbf{S}_i^2 - b(\hat{\mathbf{J}} \cdot \mathbf{S}_i)^2 + c|\hat{\mathbf{J}} \times \mathbf{S}_i|^2 \cos 2\psi_i \right], \quad (41)$$

where  $\psi_i$  is the angle subtended by the projection of  $\mathbf{S}_i$  in the plane of motion and the periastron line.

We can express these orbit averages in terms of  $x$  and  $e$  using the post-Newtonian expressions (23) and (24). Using

Eqs. (19) and (20), we find the time derivatives of the mean motion and of the eccentricity:

$$\begin{aligned} \frac{dn}{dt} = & \frac{\nu x^{11/2}}{(1-e^2)^{7/2}} \left[ \frac{1}{5} (96 + 292e^2 + 37e^4) \right. \\ & - \frac{x^{3/2}}{10(1-e^2)^{3/2}} \beta (3088 + 15528e^2 + 7026e^4 + 195e^6, 2160 + 11720e^2 + 5982e^4 + 207e^6) \\ & - \frac{x^2}{160(1-e^2)^2} \sigma (21952 + 128544e^2 + 73752e^4 + 3084e^6, 64576 + 373472e^2 + 210216e^4 + 8532e^6, \\ & 131344e^2 + 127888e^4 + 7593e^6) \\ & \left. + \frac{x^2}{320(1-e^2)^2} \tau (448 + 4256e^2 + 3864e^4 + 252e^6, 64 + 608e^2 + 552e^4 + 36e^6, 16e^2 + 80e^4 + 9e^6) \right], \quad (42) \end{aligned}$$

$$\begin{aligned} \frac{de^2}{dt} = & -\frac{\nu x^4}{(1-e^2)^{5/2}} \left[ \frac{2e^2}{15} (304 + 121e^2) - \frac{e^2 x^{3/2}}{15(1-e^2)^{3/2}} \beta (13048 + 12000e^2 + 789e^4, 9208 + 10026e^2 + 835e^4) \right. \\ & - \frac{x^2}{240(1-e^2)^2} \sigma (-320 + 101664e^2 + 116568e^4 + 9420e^6, -320 + 296672e^2 + 333624e^4 + 26820e^6, \\ & 88432e^2 + 161872e^4 + 16521e^6) \\ & \left. + \frac{x^2}{480(1-e^2)^2} \tau (-320 + 2720e^2 + 5880e^4 + 540e^6, -320 - 160e^2 + 1560e^4 + 180e^6, 16e^2 + 80e^4 + 9e^6) \right]. \quad (43) \end{aligned}$$

We find perfect agreement with [12], where the spin-orbit effects were computed in terms of  $a$  and  $e_r$ . One can worry that these derivatives depend on the angles  $\psi_i$ , which are not well-defined in the circular limit. This is however not a problem, as this dependence disappears in this limit both for  $dn/dt$  and  $de^2/dt$ .

We can see that the spin-spin couplings computed here induce a positive derivative  $de^2/dt$  for  $e \rightarrow 0$ . However, in symmetrical situations (if the projections of  $\mathbf{S}_1/m_1$  and  $\mathbf{S}_2/m_2$  on the orbital plane coincide), this derivative vanishes, due to the fact that  $\tau(1, 1, 0) - \sigma(2, 2, 0) = (P\mathbf{S}_1/m_1 - P\mathbf{S}_2/m_2)^2$ , where  $P$  is the projection operator on the orbital plane. We can compute the value of  $e^2$  for which the derivative cancels at 2PN order, which is  $e^2 = 5x^2[\tau(1, 1, 0) - \sigma(2, 2, 0)]/340$ . We emphasize that this effect is independent of the particular quasi-Keplerian parametrization one chooses (see the appendix, and in particular Eq. (A.29)).

We plotted in Fig. 1 the evolution of the eccentricity between  $x = 1/100$  and  $x = 1/6$  with spin-orbit and spin-spin couplings, for equal-mass binaries with spins uniformly distributed, including also the spin-independent PN corrections computed in [13], as well as spin-orbit precession [14]. We see that spin-orbit precession induce a non-trivial pattern in the evolution of the eccentricity, which could help to reduce the errors on spin parameters in a gravitational wave measurement. We found that the quantiles from Fig. 1 are very weakly dependent on the mass ratio, whereas the amplitudes of the modulations of the evolution of the eccentricity are strongly suppressed

as the mass ratio decreases.

## CIRCULAR LIMIT

We define the circular limit of the quasi-Keplerian motion discussed above as the limit  $e_t \rightarrow 0$ . In this limit, we also get  $e_r \rightarrow 0$  and  $e_\phi \rightarrow 0$ . The periastron line is not well defined, so that the equations of motion can only depend on differences of angles. We find (see the appendix)

$$r = x^{-1} + x^{1/2} \beta(2, 1) + \frac{x}{2} \gamma_1 - \frac{x}{2} \gamma_2 \cos[2(\phi - \psi)], \quad (44)$$

$$\frac{d\phi}{dt} = x^{3/2} - x^3 \beta(4, 3) - \frac{x^{7/2}}{2} \{3\gamma_1 + \gamma_2 \cos[2(\phi - \psi)]\}. \quad (45)$$

We note that when one includes spin-spin couplings, the orbit can no longer be circular in the sense that the radius depends explicitly on the angle along the orbit, as already mentioned in [8]. This however is not a residual eccentricity, as the radius is symmetric with respect to  $\phi \rightarrow \phi + \pi$ .

The angular frequency  $d\phi/dt$  is not constant. However, we can use Eq. (15) and define its average along an orbit

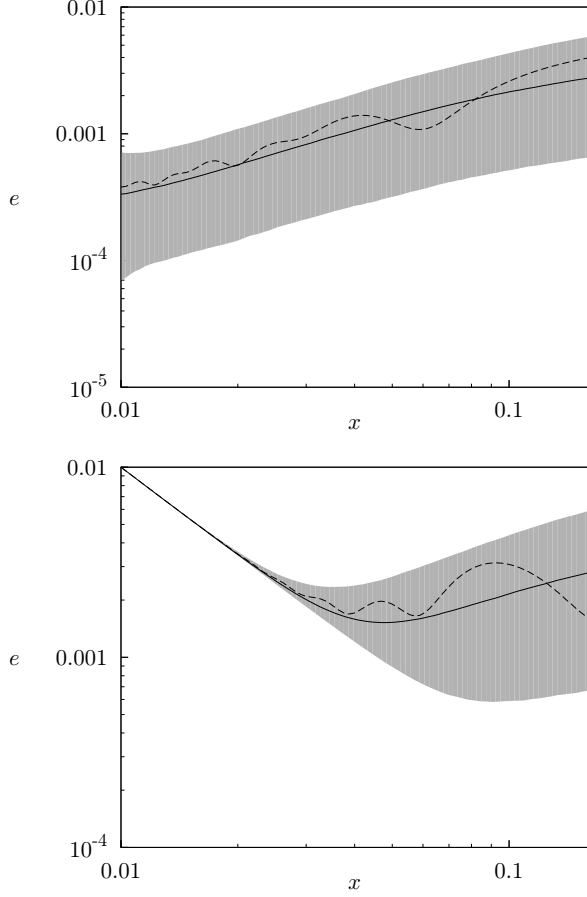


FIG. 1. Evolution of the eccentricity between  $x = 1/100$  and  $x = 1/6$  with spin-orbit and spin-spin couplings, for equal-mass binaries, at the top starting from  $e^2 = 5x^2[\tau(1,1,0) - \sigma(2,2,0)]/340$ , and at the bottom from  $e = 0.01$ , with spins uniformly distributed. In each plot, the grey region is between the 5th and the 95th percentile, the solid line is the median, and the dashed line is a typical realization.

as

$$\begin{aligned}\omega &= \frac{n}{2\pi} \int_{t(v=-\pi)}^{t(v=\pi)} \frac{d\phi}{dt} dt \\ &= \frac{n}{2\pi} [\phi(v=\pi) - \phi(v=-\pi)] \\ &= (-2E)^{3/2} \left[ 1 - \frac{1}{L^3} \beta(4,3) - \frac{3}{2L^4} \gamma_1 \right].\end{aligned}\quad (46)$$

We can thus define a new post-Newtonian parameter  $z = \omega^{2/3}$ . In terms of this parameter, the constants  $E$ ,  $L$ , and  $x$  are

$$E = -\frac{z}{2} - \frac{z^{5/2}}{3} \beta(4,3) - \frac{z^3}{2} \gamma_1, \quad (47)$$

$$L = z^{-1/2} - \frac{z}{6} \beta(20,15) - z^{3/2} \gamma_1, \quad (48)$$

$$x = z + \frac{z^{5/2}}{3} \beta(8,6) + z^3 \gamma_1. \quad (49)$$

Now, we can use Eqs. (46), (32), and (33) to find

$$\begin{aligned}\frac{d\omega}{dt} &= \frac{96\nu z^{11/2}}{5} \left[ 1 - \frac{z^{3/2}}{12} \beta(113,75) \right. \\ &\quad \left. - \frac{z^2}{48} \sigma(247,721,0) + \frac{z^2}{96} \tau(7,1,0) \right],\end{aligned}\quad (50)$$

which is in agreement with what was previously computed in [9, 15].

Alternatively, if we define a circular orbit to have  $de^2/dt = 0$ , which implies  $e^2 = 5z^2[\tau(1,1,0) - \sigma(2,2,0)]/340$ , we get

$$\begin{aligned}\frac{d\omega}{dt} &= \frac{96\nu z^{11/2}}{5} \left[ 1 - \frac{z^{3/2}}{12} \beta(113,75) \right. \\ &\quad \left. - \frac{z^2}{1216} \sigma(6519,18527,0) + \frac{z^2}{2432} \tau(439,287,0) \right].\end{aligned}\quad (51)$$

### III. CONCLUSION

The main result of this paper is the derivation of the spin-spin effects in the evolution of the mean motion and of the eccentricity, for binaries with an arbitrary eccentricity. Particularly, the fact that spin-spin couplings may induce a residual eccentricity can be important for parameter estimation when gravitational wave observations will be made possible. If eccentric templates allow to measure eccentricities of  $O(10^{-3}-10^{-4})$ , the modulation induced by spin-orbit precession could significantly improve the determination of the spins of the binary.

We also derived the equations of motion  $r(t)$  and  $\phi(t)$ , for black hole binaries of comparable mass with Newtonian, spin-orbit, and spin-spin terms on eccentric orbits, and found a family of parametrizations free of divergencies in the circular limit  $e \rightarrow 0$ .

### ACKNOWLEDGMENTS

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### Appendix: A two-parameter family of quasi-Keplerian parametrizations

We can find a two parameter family of quasi-Keplerian parametrization of the equations of motion by taking the

ansatz

$$r = a(1 - e_r \cos u) + f_{r,1} \cos(v - 2\psi) + f_{r,2} \cos(2v - 2\psi), \quad (\text{A.1})$$

$$\phi = (1 + k)v + f_{\phi,1} \sin(v - 2\psi) + f_{\phi,2} \sin(2v - 2\psi) + f_{\phi,3} \sin(3v - 2\psi), \quad (\text{A.2})$$

$$v = 2 \arctan \left( \sqrt{\frac{1 + e_\phi}{1 - e_\phi}} \tan \frac{u}{2} \right), \quad (\text{A.3})$$

$$l = n(t - t_0) = u - e_t \sin u + f_t \sin(v - 2\psi), \quad (\text{A.4})$$

where  $(r, \phi)$  is a polar coordinate system in the plane of motion,  $n$  is the mean motion,  $u$ ,  $v$ , and  $l$  are the eccentric, true, and mean anomalies,  $a$  is the semi-major axis,  $e_t$ ,  $e_r$ , and  $e_\phi$  are eccentricities,  $k$  accounts for perihelion precession, and the  $f_i$  are constants.

The constants are

$$n = (-2E)^{3/2}, \quad (\text{A.5})$$

$$a = -\frac{1}{2E} \left[ 1 - \frac{E}{L} \beta(4, 2) - \frac{E}{L^2} \gamma_1 - \lambda_1 A \frac{E}{L^2} \gamma_2 \cos 2\psi \right], \quad (\text{A.6})$$

$$k = -\frac{1}{L^3} \beta(4, 3) - \frac{3}{2L^4} \gamma_1, \quad (\text{A.7})$$

$$e_t^2 = A^2 + \frac{E}{L} \beta(8, 6 - 2A^2) + 2 \frac{E}{L^2} (\gamma_1 + A \lambda_2 \gamma_2 \cos 2\psi), \quad (\text{A.8})$$

$$e_r^2 = e_t^2 + A^2 \left[ \frac{E}{L} \beta(8, 4) + 2 \frac{E}{L^2} (\gamma_1 + A \lambda_1 \gamma_2 \cos 2\psi) \right], \quad (\text{A.9})$$

$$e_\phi^2 = e_t^2 + A^2 \left[ \frac{E}{L} \beta(8, 8) + 2 \frac{E}{L^2} (2\gamma_1 + A \lambda_1 \gamma_2 \cos 2\psi) \right], \quad (\text{A.10})$$

$$f_t = \lambda_1 \frac{(-2E)^{3/2}}{L} \gamma_2, \quad (\text{A.11})$$

$$f_{r,1} = -\frac{\lambda_2}{2} \frac{1}{L^2} \gamma_2, \quad (\text{A.12})$$

$$f_{r,2} = -\frac{1 + \lambda_1 A}{2} \frac{1}{L^2} \gamma_2, \quad (\text{A.13})$$

$$f_{\phi,1} = -\left[ A - \lambda_1 \left( 1 + \frac{3A^2}{4} \right) - \lambda_2 \right] \frac{1}{L^4} \gamma_2, \quad (\text{A.14})$$

$$f_{\phi,2} = -\frac{1 - 4A\lambda_1 - A\lambda_2}{4} \frac{1}{L^4} \gamma_2, \quad (\text{A.15})$$

$$f_{\phi,3} = \frac{\lambda_1 A^2}{4} \frac{1}{L^4} \gamma_2, \quad (\text{A.16})$$

where  $\lambda_1$  and  $\lambda_2$  are arbitrary functions of  $A$ .

The Keresztes-Mik6czi-Gergely (KMG) quasi-Keplerian parametrization of the orbit [11] is obtained by imposing  $f_{r,i} = 0$ , which leads to  $\lambda_2 = 0$  and  $\lambda_1 = -1/A$ .

To get Eqs. (14) to (17), we required that the values of the semi-major axis and of the eccentricities should

not depend on the position of the periastron line in the orbital plane. This implies  $\lambda_1 = \lambda_2 = 0$ .

We can express  $E$  and  $L$  as functions of the post-Newtonian parameter  $x = n^{2/3}$  and of the eccentricity  $e = e_t$ . We get

$$E = -\frac{x}{2}, \quad (\text{A.17})$$

$$L = \frac{\sqrt{1 - e^2}}{x^{1/2}} \left[ 1 - \frac{x^{3/2} \beta(4, 3 - e^2)}{2(1 - e^2)^{3/2}} - \frac{x^2}{2(1 - e^2)^2} (\gamma_1 + e \lambda_2 \gamma_2 \cos 2\psi) \right]. \quad (\text{A.18})$$

The constants in the equations of motion then become

$$a = x^{-1} \left[ 1 + \frac{x^{3/2} \beta(2, 1)}{\sqrt{1 - e^2}} + \frac{x^2}{2(1 - e^2)} (\gamma_1 + e \lambda_1 \gamma_2 \cos 2\psi) \right], \quad (\text{A.19})$$

$$k = -\frac{x^{3/2} \beta(4, 3)}{(1 - e^2)^{3/2}} - \frac{3x^2 \gamma_1}{2(1 - e^2)^2}, \quad (\text{A.20})$$

$$e_r = e \left[ 1 - \frac{x^{3/2} \beta(2, 1)}{\sqrt{1 - e^2}} - \frac{x^2}{2(1 - e^2)} (\gamma_1 + e \lambda_1 \gamma_2 \cos 2\psi) \right], \quad (\text{A.21})$$

$$e_\phi = e \left[ 1 - \frac{x^{3/2} \beta(2, 2)}{\sqrt{1 - e^2}} - \frac{x^2}{2(1 - e^2)} (2\gamma_1 + e \lambda_1 \gamma_2 \cos 2\psi) \right], \quad (\text{A.22})$$

$$f_t = \frac{x^2}{\sqrt{1 - e^2}} \lambda_1 \gamma_2, \quad (\text{A.23})$$

$$f_{r,1} = -\frac{x}{2(1 - e^2)} \lambda_2 \gamma_2, \quad (\text{A.24})$$

$$f_{r,2} = -\frac{x}{2(1 - e^2)} (1 + e \lambda_1) \gamma_2, \quad (\text{A.25})$$

$$f_{\phi,1} = -\frac{x^2}{(1 - e^2)^2} \left[ e - \lambda_1 \left( 1 + \frac{3e^2}{4} \right) - \lambda_2 \right] \gamma_2, \quad (\text{A.26})$$

$$f_{\phi,2} = -\frac{x^2}{(1 - e^2)^2} \left( \frac{1 - 4e\lambda_1 - e\lambda_2}{4} \right) \gamma_2, \quad (\text{A.27})$$

$$f_{\phi,3} = \frac{x^2}{(1 - e^2)^2} \frac{e^2 \lambda_1}{4} \gamma_2. \quad (\text{A.28})$$

One can see that for the solution to be free of divergencies in the zero eccentricity limit,  $\lambda_1$  and  $\lambda_2$  must be regular as  $e \rightarrow 0$ , which implies, at this PN order, that they must be regular functions of  $A$ . It is not the case for the KMG parametrization, but this is in fact a coordinate singularity, since, as we will see later, the equations of motion have a well-defined zero-eccentricity limit.

The effect of  $\lambda_1$  and  $\lambda_2$  on the evolution of the eccen-

tricity with respect to Eq. (43) is

$$\delta \frac{de^2}{dt} = \frac{\nu x^6 \lambda_2}{15(1-e^2)^{9/2}} \sigma(0, 0, 688e + 2139e^3 + 148e^5), \quad (\text{A.29})$$

which, as long as the parametrization is regular, does not affect the residual eccentricity we found in this paper.

### CIRCULAR LIMIT

Let us now compute the equations of motion  $r(\phi)$  and  $d\phi/dt$ . From Eq. (A.2), we get

$$v = (1-k)\phi - f_{\phi,1} \sin(\phi - 2\psi) - f_{\phi,2} \sin(2\phi - 2\psi) - f_{\phi,3} \sin(3\phi - 2\psi). \quad (\text{A.30})$$

Then, together with Eq. (A.3), Eq. (A.1) becomes, at

leading order in  $e$ ,

$$r = x^{-1} + x^{1/2} \beta(2, 1) + \frac{x}{2} \gamma_1 - \frac{x}{2} \gamma_2 \{ \cos[2(\phi - \psi)] + (\lambda_2 + e^2 \lambda_1) \cos(\phi - 2\psi) - e^2 \lambda_1 \cos(\phi + 2\psi) \}, \quad (\text{A.31})$$

and we can compute from Eq. (A.4), at leading order in  $e$ ,

$$\begin{aligned} \frac{d\phi}{dt} &= x^{3/2} - x^3 \beta(4, 3) - \frac{3x^{7/2}}{2} \gamma_1 - \frac{x^{7/2}}{2} \gamma_2 \cos[2(\phi - \psi)] \\ &\quad + \gamma_2 x^{7/2} [(\lambda_2 + 2e^2 \lambda_1) \cos(\phi - 2\psi) - 2e^2 \lambda_1 \cos(\phi + 2\psi)]. \end{aligned} \quad (\text{A.32})$$

One can see that the use of the KMG parametrization does not change the equations of motion in the zero eccentricity limit with respect to the one obtained from Eqs. (14) to (17). Furthermore, as the periastron line is not well-defined in the circular limit, we have to impose that  $\lambda_2 \rightarrow 0$  as  $e \rightarrow 0$  (which is equivalent to  $\lambda_2 \sim O(A)$ ), so that the equations of motion in the circular limit are independent of the choice of an arbitrary periastron line. From Eq. (A.2), we see that such an arbitrary choice induces a non-zero value for  $\phi(v=0)$  of 2PN order. However, the differences induced in the equations of motion are subsequently of 4PN order, far beyond the limit of their validity.

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